

# Fischer type determinantal inequalities for accretive-dissipative matrices

Minghua Lin

## Abstract

Let  $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$  be an  $n \times n$  accretive-dissipative matrix,  $k$  and  $l$  be the orders of  $A_{11}$  and  $A_{22}$ , respectively, and let  $m = \min\{k, l\}$ . Then

$$|\det A| \leq a |\det A_{11}| \cdot |\det A_{22}|,$$

where  $a = \begin{cases} 2^{3m/2}, & \text{if } m \leq n/3; \\ 2^{n/2}, & \text{if } n/3 < m \leq n/2. \end{cases}$  This improves a result of Ikramov.

Keywords: Accretive-dissipative matrix, Fischer determinantal inequality.

AMS subjects classification 2010: 15A45.

## 1 Introduction

Let  $\mathbb{M}_n(\mathbf{C})$  be the set of  $n \times n$  complex matrices. For any  $A \in \mathbb{M}_n(\mathbf{C})$ ,  $A^*$  stands for the conjugate transpose of  $A$ .  $A \in \mathbb{M}_n(\mathbf{C})$  is accretive-dissipative if it can be written as

$$A = B + iC, \quad (1.1)$$

where  $B = \frac{A+A^*}{2}$  and  $C = \frac{A-A^*}{2i}$  are both (Hermitian) positive definite. Conformally partition  $A, B, C$  as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} + i \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{bmatrix} \quad (1.2)$$

such that all diagonal blocks are square. Say  $k$  and  $l$  ( $k, l > 0$  and  $k + l = n$ ) the order of  $A_{11}$  and  $A_{22}$ , respectively, and let  $m = \min\{k, l\}$ .

If  $A$  is positive definite and partitioned as in (1.2), then the famous Fischer determinantal inequality (FDI) [3, p. 478] states that

$$\det A \leq \det A_{11} \cdot \det A_{22}. \quad (1.3)$$

Determinantal inequalities for accretive-dissipative matrices were first investigated by Ikramov [4], who obtained:

**Theorem 1.** *Let  $A \in \mathbb{M}_n(\mathbf{C})$  be accretive-dissipative and partitioned as in (1.2). Then*

$$|\det A| \leq 3^m |\det A_{11}| \cdot |\det A_{22}|. \quad (1.4)$$

A reverse direction to that of Theorem 1 has been given in [5]. We call this kind of inequalities the Fischer type determinantal inequality for accretive-dissipative matrices. In this paper, we intend to give an improvement of (1.4). Our main result can be stated as

**Theorem 2.** *Let  $A \in \mathbb{M}_n(\mathbf{C})$  be accretive-dissipative and partitioned as in (1.2). Then*

$$|\det A| \leq a |\det A_{11}| \cdot |\det A_{22}|, \quad (1.5)$$

$$\text{where } a = \begin{cases} 2^{3m/2}, & \text{if } m \leq n/3; \\ 2^{n/2}, & \text{if } n/3 < m \leq n/2. \end{cases}$$

As  $a < 3^m$ , it is clear that Theorem 2 improves Theorem 1. The proof of Theorem 2 is given in Section 3.

## 2 Auxiliary results

In this section, we present some lemmas that are needed in the proof of our main result.

**Lemma 3.** *[2, Property 6] Let  $A \in \mathbb{M}_n(\mathbf{C})$  be accretive-dissipative and partitioned as in (1.2). Then  $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$ , the Schur complement of  $A_{11}$  in  $A$ , is also accretive-dissipative.*

**Lemma 4.** *[4, Lemma 1] Let  $A \in \mathbb{M}_n(\mathbf{C})$  be accretive-dissipative as in (1.1). Then  $A^{-1} = E - iF$  with  $E = (B + CB^{-1}C)^{-1}$  and  $F = (C + BC^{-1}B)^{-1}$ .*

**Lemma 5.** *[7, Lemma 3.2] Let  $B, C \in \mathbb{M}_n(\mathbf{C})$  be Hermitian and assume  $B$  is positive definite. Then*

$$B + CB^{-1}C \geq 2C. \quad (2.1)$$

Here we adopt the convention that, for two Hermitian matrices  $X, Y$  of the same size,  $X > (\geq) Y$  means  $X - Y$  is positive (semi)definite. Of course, we do not distinguish  $Y < (\leq) X$  from  $X > (\geq) Y$ .

**Lemma 6.** *Let  $B, C \in \mathbb{M}_n(\mathbf{C})$  be positive semidefinite. Then*

$$|\det(B + iC)| \leq \det(B + C) \leq 2^{n/2} |\det(B + iC)|. \quad (2.2)$$

*Proof.* The first inequality follows from [6, Theorem 2.2] while the second one follows from [1, Theorem 1.1]. Here we provide a direct proof of (2.2) for the convenience of readers. We may assume  $B$  is positive definite, the general case is by a continuity

argument. Let  $\lambda_j, j = 1, \dots, n$ , be the eigenvalues of  $B^{-1/2}CB^{-1/2}$ , where  $B^{1/2}$  means the unique positive definite square root of  $B$ . Then

$$|1 + i\lambda_j| \leq 1 + \lambda_j \leq \sqrt{2}|1 + i\lambda_j|, \quad j = 1, \dots, n.$$

Also, we denote the identity matrix by  $I$ .

Compute

$$\begin{aligned} |\det(B + iC)| &= \det B \cdot |\det(I + iB^{-1/2}CB^{-1/2})| \\ &= \det B \cdot \prod_{j=1}^n |1 + i\lambda_j| \\ &\leq \det B \cdot \prod_{j=1}^n (1 + \lambda_j) \\ &= \det B \cdot \det(I + B^{-1/2}CB^{-1/2}) \\ &= \det(B + C). \end{aligned}$$

This proves the first inequality. To show the other, compute

$$\begin{aligned} \det(B + C) &= \det B \cdot \det(I + B^{-1/2}CB^{-1/2}) \\ &= \det B \cdot \prod_{j=1}^n (1 + \lambda_j) \\ &\leq \det B \cdot \prod_{j=1}^n \sqrt{2}|1 + i\lambda_j| \\ &= 2^{n/2} \det B \cdot |\det(I + iB^{-1/2}CB^{-1/2})| \\ &= 2^{n/2} |\det(B + iC)|. \end{aligned}$$

□

### 3 Main results

Theorem 2 follows from the next two theorems.

**Theorem 7.** *Let  $A \in \mathbb{M}_n(\mathbf{C})$  be accretive-dissipative and partitioned as in (1.2). Then*

$$|\det A| \leq 2^{n/2} |\det A_{11}| \cdot |\det A_{22}|. \quad (3.1)$$

*Proof.* Compute

$$\begin{aligned} |\det A| &= |\det(B + iC)| \\ &\leq \det(B + C) \quad (\text{By Lemma 6}) \\ &\leq \det(B_{11} + C_{11}) \cdot \det(B_{22} + C_{22}) \quad (\text{By FDI}) \\ &\leq 2^{k/2} |\det(B_{11} + iC_{11})| \cdot 2^{l/2} |\det(B_{22} + iC_{22})| \quad (\text{By Lemma 6))} \\ &= 2^{n/2} |\det A_{11}| \cdot |\det A_{22}|. \end{aligned}$$

□

**Theorem 8.** *Let  $A \in \mathbb{M}_n(\mathbf{C})$  be accretive-dissipative and partitioned as in (1.2). Then*

$$|\det A| \leq 2^{3m/2} |\det A_{11}| \cdot |\det A_{22}|. \quad (3.2)$$

*Proof.* We have, by Lemma 4, that

$$\begin{aligned} A/A_{11} &= A_{22} - A_{21}A_{11}^{-1}A_{12} \\ &= B_{22} + iC_{22} - (B_{12}^* + iC_{12}^*)(B_{11} + iC_{11})^{-1}(B_{12} + iC_{12}) \\ &= B_{22} + iC_{22} - (B_{12}^* + iC_{12}^*)(E_k - iF_k)(B_{12} + iC_{12}) \end{aligned}$$

with

$$E_k = (B_{11} + C_{11}B_{11}^{-1}C_{11})^{-1}, \quad F_k = (C_{11} + B_{11}C_{11}^{-1}B_{11})^{-1}.$$

By Lemma 5 and the operator reverse monotonicity of the inverse, we get

$$E_k \leq \frac{1}{2}C_{11}^{-1}, \quad F_k \leq \frac{1}{2}B_{11}^{-1}. \quad (3.3)$$

Setting  $A/A_{11} = R + iS$  with  $R = R^*$  and  $S = S^*$ . By Lemma 3, we know  $R$  and  $S$  are positive definite. A calculation shows

$$\begin{aligned} R &= B_{22} - B_{12}^*E_kB_{12} + C_{12}^*E_kC_{12} - B_{12}^*F_kC_{12} - C_{12}^*F_kB_{12}; \\ S &= C_{22} + B_{12}^*F_kB_{12} - C_{12}^*F_kC_{12} - C_{12}^*E_kB_{12} - B_{12}^*E_kC_{12}. \end{aligned}$$

It can be verified that

$$\begin{aligned} \pm(B_{12}^*F_kC_{12} + C_{12}^*F_kB_{12}) &\leq B_{12}^*F_kB_{12} + C_{12}^*F_kC_{12}; \\ \pm(C_{12}^*E_kB_{12} + B_{12}^*E_kC_{12}) &\leq B_{12}^*E_kB_{12} + C_{12}^*E_kC_{12}. \end{aligned}$$

Thus,

$$R + S \leq B_{22} + 2B_{12}^*F_kB_{12} + C_{22} + 2C_{12}^*E_kC_{12}. \quad (3.4)$$

As  $B, C$  are positive definite, we also have

$$B_{22} > B_{12}^*B_{11}^{-1}B_{12}, \quad \text{and} \quad C_{22} > C_{12}^*C_{11}^{-1}C_{12}. \quad (3.5)$$

Without loss of generality, we assume  $m = l$ . Compute

$$\begin{aligned} |\det(A/A_{11})| &= |\det(R + iS)| \\ &\leq \det(R + S) \quad (\text{by Lemma 6}) \\ &\leq \det(B_{22} + 2B_{12}^*F_kB_{12} + C_{22} + 2C_{12}^*E_kC_{12}) \quad (\text{by (3.4)}) \\ &\leq \det(B_{22} + B_{12}^*B_{11}^{-1}B_{12} + C_{22} + C_{12}^*C_{11}^{-1}C_{12}) \quad (\text{by (3.5)}) \\ &< \det(2(B_{22} + C_{22})) \quad (\text{by (3.5)}) \\ &= 2^m \det(B_{22} + C_{22}) \\ &\leq 2^m \cdot 2^{m/2} |\det(B_{22} + iC_{22})| \quad (\text{by Lemma 6}) \\ &= 2^{3m/2} |\det A_{22}|. \end{aligned}$$

The proof is complete by noting  $\det(A/A_{11}) = \frac{\det A}{\det A_{11}}$ . □

It is natural to ask whether  $a$  in (1.5) can be replaced by a smaller number? There is evidence that the following could hold:

**Conjecture 9.** *Let  $A \in \mathbb{M}_n(\mathbb{C})$  be accretive-dissipative and partitioned as in (1.2). Then*

$$|\det A| \leq 2^m |\det A_{11}| \cdot |\det A_{22}|.$$

We end the paper by an example showing that if the above conjecture is true, then the factor  $2^m$  is optimal.

**Example 10.** *Let  $A = \begin{bmatrix} (1+\epsilon)(1+i) & i-1 \\ i-1 & (1+\epsilon)(1+i) \end{bmatrix} = \begin{bmatrix} 1+\epsilon & -1 \\ -1 & 1+\epsilon \end{bmatrix} + i \begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1+\epsilon \end{bmatrix}$  with  $\epsilon > 0$ . Then  $A$  is accretive-dissipative. As  $\epsilon \rightarrow 0^+$ , we have*

$$\frac{|\det A|}{|\det A_{11}| \cdot |\det A_{22}|} \rightarrow 2.$$

## 4 Acknowledgement

The author thanks the referee for his/her careful reading of the manuscript.

## References

- [1] R. Bhatia, F. Kittaneh, The singular values of  $A + B$  and  $A + iB$ , Linear Algebra Appl. 431 (2009) 1502-1508.
- [2] A. George, Kh. D. Ikramov, On the properties of accretive-dissipative matrices, Math. Notes 77 (2005) 767-776.
- [3] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, London, 1985.
- [4] Kh. D. Ikramov, Determinantal inequalities for accretive-dissipative matrices, J. Math. Sci. (N. Y.) 121 (2004) 2458-2464.
- [5] M. Lin, Reversed determinantal inequalities for accretive-dissipative matrices, Math. Inequal. Appl. 12 (2012) 955-958.
- [6] X. Zhan, Singular values of differences of positive semidefinite matrices, SIAM J. Matrix Anal. Appl. 22 (2000) 819-823.
- [7] X. Zhan, Computing the extremal positive definite solutions of a matrix equation, SIAM J. Sci. Comput. 17 (1996) 1167-1174.

Minghua Lin

Department of Applied Mathematics,  
University of Waterloo,  
Waterloo, ON, N2L 3G1, Canada.  
mlin87@ymail.com